JUST WHAT IS ALGEBRAIC THINKING?

By Shelley Kriegler

The goal of “algebra for all” has been in place in this country for more than a decade, driven by the need for quantitatively literate citizens and a recognition that algebra is a gatekeeper to more advanced mathematics and opportunities (Silver, 1997; Dudley, 1997). To accomplish this goal, many states, including California, have established algebra as its grade level course for eighth graders (California Board of Education, 1997) . Unfortunately, the data clearly show that all students are not succeeding in algebra in the eighth grade. For example, in 2006 only 22% of California’s eighth graders demonstrated proficiency in a course equivalent to algebra or higher (Kriegler & Lee, 2007). The implication is clear: elementary and middle school mathematics instruction must focus greater attention on preparing all students for challenging middle and high school mathematics programs that include algebra (Chambers, 1994; Silver, 1997). Thus, “algebraic thinking” has become a catch-all phrase for the mathematics teaching and learning that will prepare students with the critical thinking skills needed to fully participate in our democratic society and for successful experiences in algebra.

COMPONENTS OF ALGEBRAIC THINKING

In this article, algebraic thinking is organized into two major components: the development of mathematical thinking tools and the study of fundamental algebraic ideas. Mathematical thinking tools are analytical habits of mind. They are organized around three topics: problem-solving skills, representation skills, and quantitative reasoning skills. Fundamental algebraic ideas represent the content domain in which mathematical thinking tools develop. They are explored through three lenses: algebra as generalized arithmetic, algebra as a language, and algebra as a tool for functions and mathematical modeling. *Figure 1* (on the next page) summarizes these components and gives citations to the 2006 California Mathematics Content Standards (California Board of Education, 2006). These citations illustrate a consistency between this algebraic thinking framework and a carefully crafted state mathematics standards document.

Within the algebraic thinking framework outlined here, it is easy to understand why lively discussions occur within the mathematics community regarding what mathematics should be taught and how. Those who argue that the study of mathematics is important because it helps to develop logical processes may consider mathematical thinking tools as the more critical component of mathematics instruction. On the other hand, those who express concern about the lack of content and rigor within the discipline itself may be focusing greater emphasis on the algebraic ideas themselves. In reality, both are important. One can hardly imagine thinking logically (mathematical thinking tools) with nothing to think about (algebraic ideas). On the other hand, algebra skills that are not understood or connected in logical ways by the learner remain “factoids” of information that are unlikely to increase true mathematical competence.

Mathematical Thinking Tools

Figure 1: **Components of Algebraic Thinking (with illustrative citations from the 2006 *Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve*)**

Mathematical Thinking Tools	**Fundamental Algebraic Ideas**
Problem solving skills • Using problem solving strategies (Gr 7 MR 1.3, MR 2.0). • Exploring multiple approaches/multiple solutions (Gr 7 MR 2.0, 2.5).	Algebra as generalized arithmetic • Conceptually based computational strategies (Gr 7 NS 2.2). • Ratio and proportion (Gr 6 NS 1.2, 1.3). • Estimation (Gr 7 MR 2.7).
Representation skills • Displaying relationships visually, symbolically, numerically, verbally (Gr 7 MR 2.5). • Translating among different representations (Gr 7 MR 2.3, 2.5). • Interpreting information within representations (Gr 7 MR 2.5, 2.6). Quantitative reasoning skills • Analyzing problems to extract and quantify essential features (Gr 7 MR 1.1). • Inductive and deductive reasoning (Gr 7 MR 2.4).	Algebra as the language of mathematics • Meaning of variables and variable expressions (Gr 4 AF 1.1). • Meaning of solutions (Gr 4 NS 1.4). • Understanding and using properties of the number system (Gr 5 AF 1.3). • Reading, writing, manipulating numbers and symbols using algebraic conventions (Gr 6 AF 1.3). • Using equivalent symbolic representations to manipulate formulas, expressions, equations, inequalities (Gr 4 AF 2.1, 2.2; Gr 7 AF 1.3). Algebra as a tool for functions and mathematical modeling • Seeking, expressing, generalizing patterns and rules in real-world contexts (Gr 6 AF 3.2). • Representing mathematical ideas using equations, tables, graphs, or words (Gr 6 AF 1.0). • Working with input/output patterns (Gr 4 AF 1.5). • Developing coordinate graphing skills (Gr 5 AF 1.4).

Mathematical thinking tools are organized here into three general categories: problem-solving skills, representation skills, and reasoning skills. These thinking tools are essential in many subject areas, including mathematics; and quantitatively literate citizens utilize them on a regular basis in the workplace and as part of daily living.

Problem-solving requires having the mathematical tools to figure out what to do when one does not know what to do. Students who have a toolkit of problem-solving strategies (e.g., guess and check, make a list, work backwards, use a model, solve a simpler problem, etc.) are better able to get started on a problem, attack the problem, and figure out what to do with it. Furthermore, since the real world does not include an answer key, exploring mathematical problems using multiple approaches or devising mathematical problems that have multiple solutions gives students opportunities to develop good problem-solving skills and experience the utility of mathematics.

The ability to make connections among multiple representations of mathematical information gives students quantitative communication tools. Mathematical relationships can be displayed in many forms including visually (i.e. diagrams, pictures, or graphs), numerically (i.e. tables, lists, with computations), symbolically, and verbally[1]. Often a good mathematical explanation includes several of these representations because each one contributes to the understanding of the ideas presented. The ability to create, interpret, and translate among representations gives students powerful tools for mathematical thinking.

Finally, quantitative reasoning is fundamental to success in mathematics, and algebraic thinking helps develop quantitative reasoning within an algebraic framework (Kieran and Chalouh, 1993). Since applications of mathematics rarely require making calculations on "naked numbers," analyzing problems to extract and quantify relevant information is an essential reasoning skill. Inductive reasoning involves examining particular cases, identifying patterns and relationships among those cases, and extending the patterns and relationships. Deductive reasoning involves drawing conclusions by examining a problem's structure. Quantitatively literate citizens routinely utilize these types of reasoning on a regular basis.

Fundamental Algebraic Ideas

The line between the study of informal algebraic ideas and formal algebra is often blurred, and the algebra ideas identified here are intended to be studied in concrete or familiar contexts so that students will develop a strong conceptual base for later abstract study of mathematics. In this framework, algebraic ideas are viewed in three ways: algebra as generalized arithmetic, algebra as a language, and algebra as a tool for functions and mathematical modeling.

Algebra is sometimes referred to as generalized arithmetic; therefore, it is essential that instruction give students opportunities to make sense of general procedures performed on numbers and quantities (Battista and Van Auken Borrow, 1998; Vance, 1998). According to Battista, thinking about numerical procedures should begin in the elementary grades and continue until students can eventually express and reflect on procedures using algebraic symbol manipulation (1998). By routinely encouraging conceptual approaches when studying arithmetic procedures, students will develop a network of mathematical structures to draw upon when they begin their study of formal algebra. Here are three examples:

- Elementary school children typically learn to multiply whole numbers using the "U.S. Standard Algorithm." This procedure is efficient, but the algorithm easily obscures important mathematical connections, such as the role of the distributive property in multiplication or how area and multiplication are connected. These require attention as well.

- The "means-extremes" procedure for solving proportions provides middle school students with an easy-to-learn rule, but does little to help them understand the role of the multiplication property of equality in solving equations or develop sense-making notions about proportionality. These ideas are essential to the study of algebra, and attention to their conceptual development will ease the transition to a more formal study of the subject.

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[1] Kriegler et al. (2007) use this approach extensively in an algebra readiness program called *Introduction to Algebra*. It is called the "fourfold way."

- The widely accepted distance from the earth to the sun is estimated at 93 million miles, but establishing some referants for the meaning of the magnitude of 93,000,000 requires manipulation of ratios and rates and a well-developed generalized number sense.

Algebra is a language (Usiskin, 1997). To comprehend this language, one must understand the concept of a variable and variable expressions, and the meanings of solutions. It involves appropriate use of the properties of the number system. It requires the ability to read, write, and manipulate both numbers and symbolic representations in formulas, expressions, equations, and inequalities. In short, being fluent in the language of algebra requires understanding the meaning of its vocabulary (i.e. symbols and variables) and flexibility to use its grammar rules (i.e. mathematical properties and conventions). Historically, beginning algebra courses have emphasized this view of algebra. Here are two examples:

- How to interpret symbols or numbers that are written next to each other can be problematic for students. In our number system, the symbol “149” means “one hundred forty-nine.” However, in the language of algebra, the expression “14*x*” means “multiply fourteen by ‘*x*.’” Furthermore, $x14 = 14x$, but “14*x*” is the preferred expression because, by convention, we write the numeral or “coefficient” first.

- The variables used in algebra take on different meanings, depending on context. For example, in the equation $3 + x = 7$, “*x*” is an unknown, and “4” is the solution to the equation. But in the statement $A(x + y) = Ax + Ay$, the “*x*” is being used to generalize a pattern.

Finally, algebra can be viewed as a tool for functions and mathematical modeling. Through this lens, algebraic thinking shows students the real-life uses and relevance of algebra (Herbert and Brown, 1997). Seeking, expressing, and generalizing patterns and rules in real world contexts; representing mathematical ideas using equations, tables, and graphs; working with input and output patterns; and developing coordinate graphing techniques are mathematical activities that build algebra-related skills. Functions and mathematical modeling represent contexts for the application of these algebraic ideas.

EXAMPLES OF ALGEBRAIC THINKING

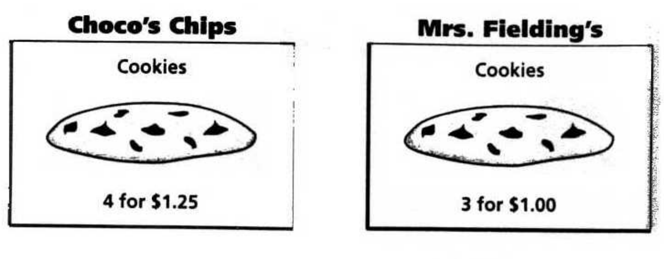
Two problems and their solutions (see *Figure 2*) are used here to illustrate some of the decribed components of algebraic thinking. “Smart Shopping” (Greenes and Findell, 1998) is a problem that can lead to generalized thinking about arithmetic. “The Garden Problem” (Creative Publications, 1998) requires students to find an algebraic expression for a geometric figure and exemplifies algebra as a language, and as a tool for functions and modeling. Furthermore, many mathematical thinking tools are evident within the student solutions. They use diverse approaches and solution strategies, they communicate ideas in a variety of ways, and they use explanations to show evidence of analytical reasoning.

Neither of these problems is likely to be difficult for people with well-developed algebra skills, and both of these problems can be solved without using much algebraic thinking. For example, most students initially answered the “Smart Shopping” problem by computing unit prices (see

Figure 3 - Solution 1), and many students initially found the number of tiles required for a garden with a length of 12 units by drawing a picture and counting (see *Figure 4* - Solution 6). However, challenging students to find solutions using more than one method created practice opportunities for algebraic thinking. By sharing teacher and student methods in class, students began to adopt the mathematical thinking tools and algebra skills of others.

Figure 2: **Two Problems**

"Smart Shopping"

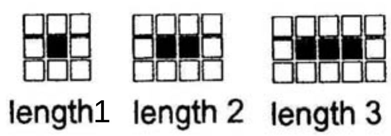


Two shops sell chocolate chip cookies.

A. Kelly wants to buy cookies. Which shop has the better buy?
B. Explain your answer.

"The Garden Problem"

Gardens are framed with a single row of tiles as illustrated here.
(A garden of length 3 requires 12 border tiles.)



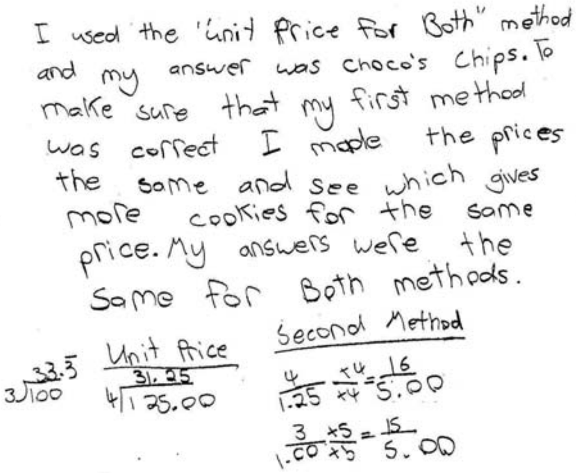
A. How many border tiles are required for a garden of length 12?
B. How many border tiles are required for a garden of length "*n*"?
C. Show how to find the length of the garden if 152 tiles are used for the border.

Mathematical Thinking Tools Revisited

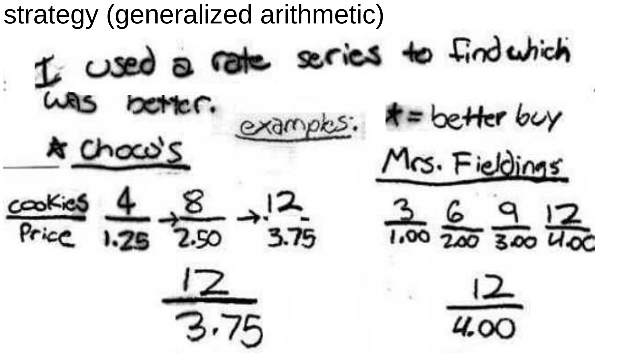
Mathematical thinking tools were used in a variety of ways when students solved these problems. Problem-solving approaches included making a table, looking for patterns, using models and diagrams, and working backwards. Students represented solutions numerically, symbolically, graphically, and verbally. They extracted relevant numerical information from the problems, and their explanations provided evidence of both inductive and deductive reasoning. For example, in *Figure 4* - Solution 7, the student used specific cases in the table to predict a numerical pattern. In *Figure 6* - Solution 16, the student used the structure of the problem to create an inverse function, and expressed it with symbols.

Figure 3: **Solution to "Smart Shopping" – Part A (Which is a better buy?)**

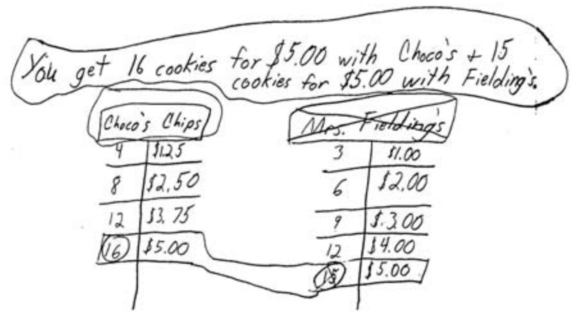
Solution 1: Uses multiplicative identity in second method (generalized arithmetic)



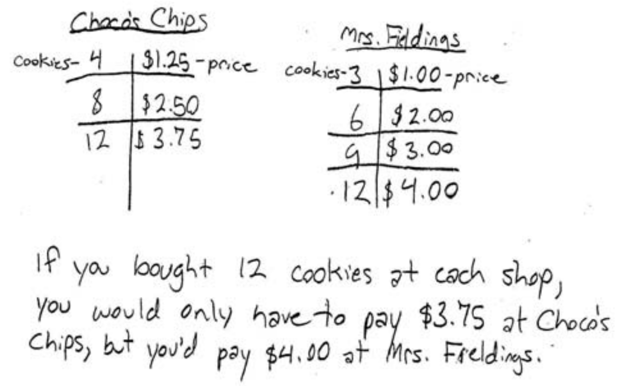
Solution 2: Uses conceptually based computation strategy (generalized arithmetic)



Solution 3: Uses proportional thinking strategy (generalized arithmetic) and interprets an input-output pattern (function)



Solution 4: Uses proportional thinking strategy (generalized arithmetic) and interprets input-output pattern (function)



Solution 5: Represents and interprets mathematical idea using a coordinate graph (function)

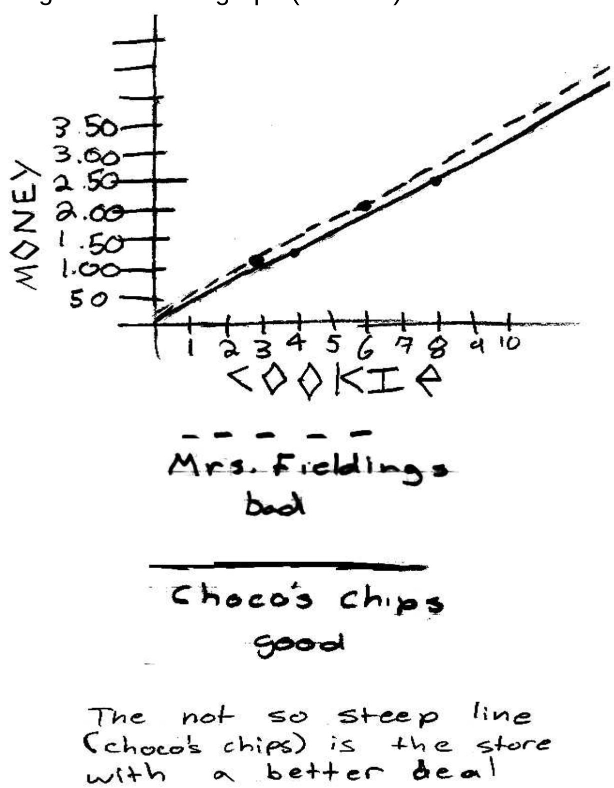
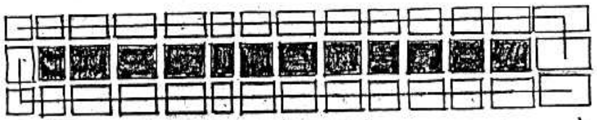


Figure 4: **Solutions To "The Garden Problem" – Part A**
How many border tiles are required for a garden of length 12?

Solution 6: Begins to seek and express a pattern for finding perimeter tiles (functions/modeling)



14 on top & Bottom & 1 on each side

14+14+1+1

Solution 7: Seeks pattern for number of tiles needed, represents relationship with input-output table (functions/modeling)

length	tiles
1	8
2	10
3	12
4	14
5	16
6	18
7	20
8	22
9	24
10	26
11	28
12	30

The frame gets bigger by two every time the garden gets bigger by one.

Solution 8: Identifies general way to express perimeter of garden (functions/modeling), uses associative and commutative properties to compute (generalized arithmetic, language)

top + side + bottom + side

12+3+12+3

(12+12) + (3+3)

24 + 6

30

Solution 9: Describes general pattern for finding perimeter tiles (functions/modeling), uses distributive property to compute (generalized arithmetic, language)

3×14= 3×4+3×10

3×14= 12+30

3×14=42

42-12=30

I found the area of the whole garden is 14×3 So when I found it to be 42, I still had to subtract the inside which is 12. My answer is 30.

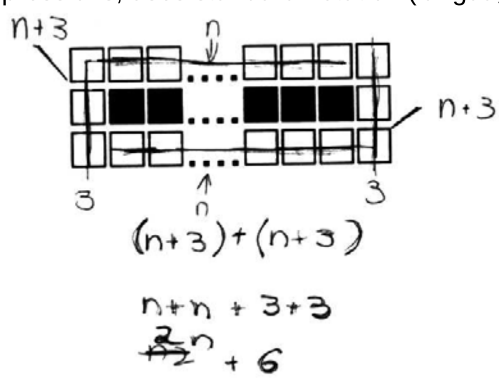
More About Algebraic Ideas

Student solutions to “Smart Shopping” (*Figure 3*) illustrate several informal algebraic ideas. Use of the multiplicative identity to find equivalent fractions is evident in the second method of Solution 1. Although it is unclear whether students used additive or multiplicative procedures, the potential for proportional reasoning is clearly demonstrated in the rate series (Solution 2) and in the tables (Solutions 3 and 4). Some students used function concepts to solve the “Smart Shopping” problem. In Solutions 3 and 4, students analyzed the inputs and outputs in their tables to arrive at conclusions. In Solution 5, the student used a rate graph to solve the problem.

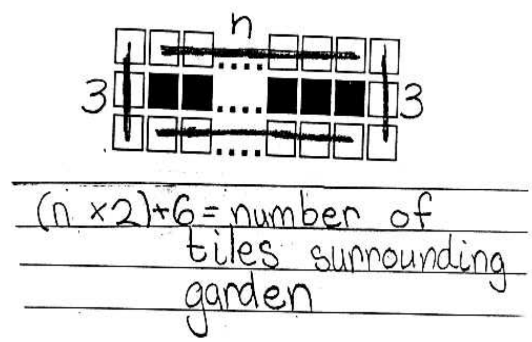
Student work from the three parts of the “The Garden Problem” also provide evidence of many informal algebraic ideas that are important in the development of algebraic thinking (see *Figures 4, 5, 6*). In Part A, students used conceptual approaches to arithmetic, which helped them to understand the pattern within the problem. Part B required the use of a variable to represent an algebraic expression. Students used deductive thinking in both Parts B and C to find and apply functional relationships for the number of tiles in the geometric design.

Figure 5: **Solutions to “The Garden Problem” – Part B**
How many border tiles are required for a garden of length “*n*”?

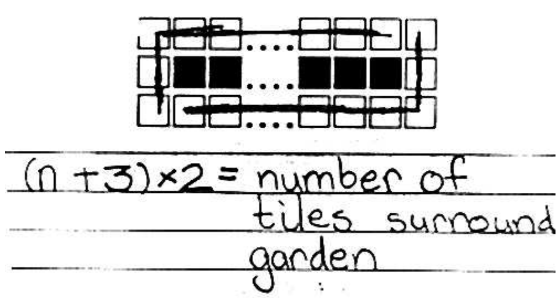
Solution 10: Generalizes pattern with symbolic rule (functions/modeling), finds equivalent algebraic expressions, uses standard notation (language)



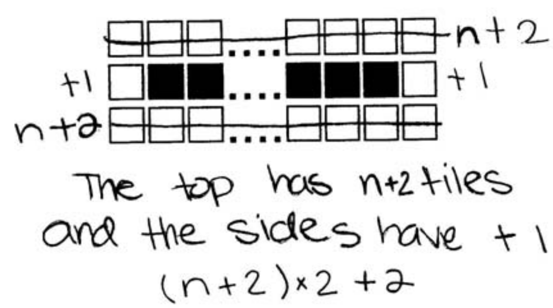
Solution 12: Generalizes pattern with symbolic rule (functions/modeling), uses variable in expression (language)



Solution 11: Generalizes pattern with symbolic rule (functions/modeling), uses variable in expression (language)

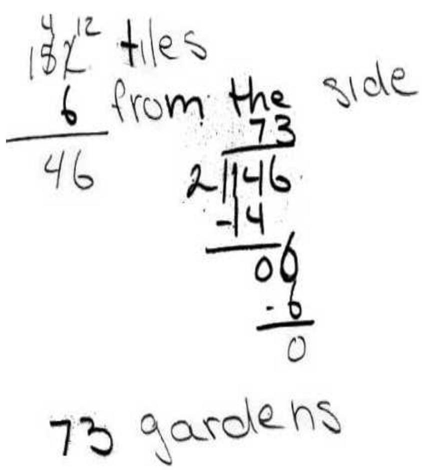
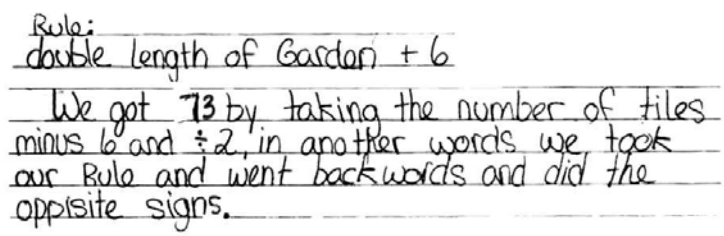
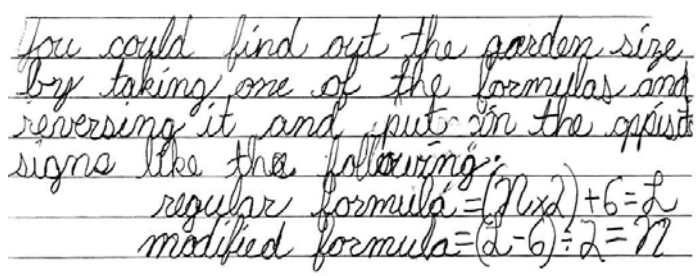


Solution 13: Generalizes pattern with symbolic rule (functions/modeling), uses variable in expression (language)



Finding the answer to Part A *(Figure 4)* was not difficult for students; however, doing it in a way that revealed something about the structure of the geometric design laid the groundwork for a formula or expression. In Solution 6, the student added the top and bottom of the border to the tiles on each side. This solution can lead to general expressions such as $(n+2)+(n+2)+1+1$ or $2(n+2)+2$. The author of Solution 7 made a table and explained that each stage of the garden design increased by 2. This approach helps to explain the need to multiply the garden length by 2 in a general expression such as $2n+6$. Solutions 8 and 9 revealed interesting breakdowns of the parts of the garden, and the computation procedures illustrated understanding of mathematical properties important to algebra. In Solution 8, the student used both the associative and commutative properties as she added the number of tiles above and below the garden to those on the sides. The student who wrote Solution 9 viewed the garden plot as a solid rectangle and then subtracted the garden itself. The distributive property is implied within her calculation: $3(14) = 3(10+4) = 3(10)+3(4)$.

Figure 6: **Solutions to "The Garden Problem" – Part C**
Show how to find the length of the garden if 152 tiles are used for the border.

<table>
<tr>
<td rowspan="2">Solution 14: Given output, calculates input (function)

</td>
<td>Solution 15: Informally describes an output-input relationship (function)

</td>
</tr>
<tr>
<td>Solution 16: Manipulates formulas (language), uses output-input relationship (function)

</td>
</tr>
</table>

Part B (*Figure 5*) required the use of a variable to express the number of tiles needed to border the garden. Student solutions reveal four equivalent expressions derived directly from examining patterns in the garden, and the solutions show that the students' abilities to use established conventions for writing expressions are developing. For example, in Solution 10, the student showed that he understood that "$n+n$" is equal to "n times 2," which can be written $n2$ or $2n$. By his work, we see that he is learning that it is customary to write the numerical coefficient first.

Finding multiple expressions for the number of tiles in the border can lead to other algebraic thinking opportunities. For example, from Solutions 10 and 11, we see that $2(n + 3) = 2n + 6$. Verifying that the symbolic expressions are equivalent creates practice for simplifying expressions and a context for discussion of mathematical properties. Substituting values into the expressions to find the number of border tiles needed for specific cases helps students to understand the meaning of solutions and practice using order of operations.

Solutions to Part C (*Figure 6*) show students' emerging abilities to find and use an inverse function. In Solution 14, the student's computational procedure demonstrates her ability to apply an inverse process for a specific case. In Solution 15, the student explained verbally how the rule must be applied "backwards." The author of Solution 16 was able to write both the function and inverse function symbolically.

CONCLUSION

As large percentages of students struggle to prepare for algebra, programs that support student development of algebraic thinking are becoming increasingly visible in schools. California, for example, recognized the need to offer such programs, and included algebra readiness standards in its 2007 *Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve.* Curricular materials such as *Introduction to Algebra* (Kriegler et al., 2007) have been developed to meet these standards.

Algebra is recognized as a gateway to higher education and opportunities, and successful participation in our democratic society and technology-driven world requires the abstract mathematical thinking inherent in it (Dudley, 1998; Riley, 1998). But for students to achieve access to and success in the formal study of algebra, they need to achieve fluency using mathematical thinking tools and informal algebraic ideas. This research-based algebraic thinking framework can be used to critically examine goals, materials, and instruction strategies that promote the development of mathematical thinking tools and fundamental algebraic ideas, and serve as a reminder of important elements in the development of algebraic thinking.

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